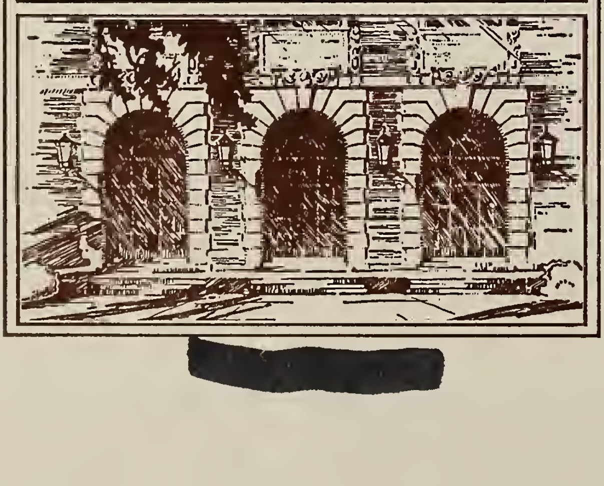

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STABILITY AND CONVERGENCE OF VARIABLE ORDER MULTISTEP METHODS

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1. INTRODUCTION

This paper is a sequel to a paper by Gear and Tu [1] which discussed the stability and convergence of variable step multistep methods. A familiarity with that paper is assumed. Just as the stability and convergence of multistep methods depend on the step selection scheme they depend on the order selection scheme. This paper considers conditions under which order selection schemes do not make a constant order stable multistep method unstable. It is shown that the interpolation step changing technique used with Adams formula is stable provided that both the step and order remain constant for r steps after any change, where r-1 is the current order of the method. The variable step technique for changing step sizes used with Adams methods is shown to be stable with respect to any order and step selection schemes. In fact, this is shown to be true for all "constant-ρ" methods; that is, ones in which the corrector coefficients $(\alpha_1^*, \ldots, \alpha_k^*)$ are independent of the step and order selection schemes.

Although the most common reason for switching from one formula to another is to change order, there can be other reasons, e.g. one might switch between Adams and stiff formulas of the same order because of a change in the stiffness of the system. Consequently, we discuss "variable formula methods" rather than just variable order methods.

In [1] it was shown that a step selection scheme which produced step size changes small in the sense that

$$\frac{h_{n+1}}{h_n} = 1 + O(h)$$

gave stable, convergent interpolation or variable step methods if the underlying multistep formula was stable. This result is extended to show that for any class of strongly stable formulas $\mathcal{F} = \{F_i | i = 1, 2, \ldots\}$, there exist constants p_i such that both the interpolation and variable step methods based on $\mathcal{F}$ are stable with respect to small step changes provided that the formula selection scheme guarantees at least p_i steps are used without change once a method based on F_i is selected. This result is demonstrated to be sharp in the sense that classes of weakly stable formulas may give instability, and that the p_i have to be larger than one for some classes of strongly stable formulas.

2. BACKGROUND

To simplify the presentation, single number references are to the paper [1]. Numbers of all definitions, equations, theorems and lemmas in this paper are prefixed by 2. Thus, Theorem 2.3 is the third theorem in this paper, while equation (17) is in reference [1].

The way in which variable step and interpolation techniques are used to generate methods M_i based on formulas F_i was previously discussed in [1]. In a variable order method, we have a set of formulas $\{F_i | i = 1, 2, \ldots\}$ (possibly of different orders) with an associated set of methods M_i which depend on the step changing technique used. Changing from one method M_i to another M_j may require a special operation O_{ji}. Hence, we state:

<u>*Definition 2.1*</u>

A variable formula method is a set of multistep methods, $\{M_i\}$, with underlying formulas, $\{F_i\}$, and operations for changing, $\{O_{ij}\}$.

<u>*Definition 2.2*</u>

A formula selection scheme I is a function $I(h,t)$ such that the formula used for the interval $[t_n, t_{n+1}]$ is $F_{I(h,t_n)}$, where h is the controlling parameter in the step selection scheme (see definition 1).

<u>Examples</u>

A1. Suppose the set of formulas is the set of i-step explicit Adams-Bashforth formulas F_i, i = 1, 2,..., k, where k is the maximum number of

steps ever used. If the variable step technique is used for changing step sizes, then the method is given by

$$y_{n+1} = y_n + \beta_{1,n} h_{n-1} y'_n + \ldots + \beta_{k,n} h_{n-k} y'_{n-k+1},$$

where if a method based on F_i is in use, the $\beta_{j,n}$ are determined by

$$\beta_{j,n} = 0, \quad j > i,$$

and the requirement that the method have order i. No special action is needed to change order. The vector of saved values can formally be defined to include k past values of the derivative by

$$\underline{y}_n = [y_n, h_{n-1} y'_n, \ldots, h_{n-k} y'_{n-k+1}]^T,$$

and the method can be represented by the matrix

$$A_n = \begin{bmatrix} 1 & \beta_{1,n} & \beta_{2,n} & \cdots & \beta_{k-1,n} & \beta_{k,n} \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ & \cdots & & \cdots & \cdots & \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix},$$

and the vector

$$\underline{\ell}_n = [0, 1, 0, \ldots, 0]^T.$$

The method changing operations are represented as matrices, in this case by the identity, so $0_{ij} = I$.

A2. If the Nordsieck form of Adams method is used, and step changing is handled by interpolation, we have from equations (9) through (12)

$$\underline{y}_n = [y_n, h_{n-1} y_n', \ldots, h_{n-1}^k y_n^{(k)}/k!]^T,$$

$$A_n = A\, C_n,$$

where

$$A = \left[\begin{array}{ccccc|c} 1 & 1 & 1 & \ldots & 1 & \\ & 1 & 2 & \ldots & i & \\ & & & \ldots & & 0 \\ & & & & 1 & \\ \hline & 0 & & & & 0 \end{array}\right]$$

if the method is currently equivalent to an i-step method,

$$C_n = \text{diag}[1, \frac{h_n}{h_{n-1}}, (\frac{h_n}{h_{n-1}})^2, \ldots, (\frac{h_n}{h_{n-1}})^k],$$

and $\underline{\ell}_n$ is the vector for an i-step method extended to k+1 entries with zeros. In this case, changing the order may involve some operations. Decreasing the order can be done by dropping the higher order derivatives. Since they are ignored in lower order methods, we can take $O_{ij} = I$ if $i \leq j$. When the order is increased, we could elect to start with zeros as approximations to the new derivatives required in which case we could take $O_{ij} = I$ also for $i > j$. Another common way of increasing order is to only allow an order increase of one at a time, and, if the order has just increased to j, estimate $h^j y^{(j)}/j!$ by computing 1/j times the backward difference of $h^{j-1} y^{(j-1)}/(j-1)!$. To do this, we need to save the previous value of the latter scaled derivative, so we must modify A_n to be

$$A_n = \left[\begin{array}{ccccc|c} 1 & 1 & 1 & \cdots & 1 & \\ & 1 & 2 & \cdots & i & \\ & & 1 & & & 0 \\ & & & \ddots & & \\ & 0 & & & 1 & \\ \hline & & & & 1 & \\ & & & & & 0 \\ & & & & & \end{array}\right] \tag{2.1}$$

when an i-th order method is being used and $i < k$.

Then

$$O_{i+1,i} = \begin{array}{c} \left[\begin{array}{ccccccc} 1 & & & & & & \\ & \ddots & & & 0 & & \\ & & 1 & & & & \\ & & \frac{1}{i+1} & \frac{-1}{i+1} & & & \\ & & & & 1 & & \\ & 0 & & & & \ddots & \\ & & & & & & 1 \end{array}\right] \end{array} \begin{array}{l} \\ \\ (i+1)\text{-th row} \\ (i+2)\text{-th row} \\ \\ \\ \\ \end{array} \tag{2.2}$$

produces the required difference since the $(i+2)$-th row is used to save the previous value of the i-th scaled derivative when the order i is less than k. We will call this the difference method of changing order.

Difference between examples A1 and A2

Since the Nordsieck representation is a transformation of the multistep representation, there do exist matrices O_{ij} for the Nordsieck representation such that changing the order in the latter representation

is equivalent to the method used in Example A1. However, these matrices are complex (a discussion of what is needed for Adams methods can be found in Shampine and Gordon [2]). A representation is chosen with a view to making computations simple. Therefore, we expect that if the Nordsieck vector is used, the interpolation step changing and order changing techniques discussed in A2 will be used, whereas if a multistep representation is used, a variable step technique and the type of order changing discussed in A1 will be used. For this reason, we will talk about order changing with variable step and interpolation step changing techniques, meaning that the type of order changing discussed in Examples A1 and A2 are used respectively.

2.1 Stability and Convergence

When the formula changing process is represented by a matrix, as above, we can derive an error equation similar to equation (20). It will be

$$\underline{\varepsilon}_{n+1} = S_n \, O_{i_{n+1} i_n} \, \underline{\varepsilon}_n + \underline{d}_n ,$$

where i_n is the number of the formula used on the step from t_{n-1} to t_n. To simplify the notation, we will write O_n for $O_{i_{n+1} i_n}$, and $T_n = S_n O_n$.

Stability and convergence were defined in definitions 2 and 3 with respect to a step selection scheme θ. For variable formula methods we extend the definitions in the obvious way with respect to formula and step selection schemes.

Definition 2.3

A method is stable with respect to a step selection scheme θ *and a formula selection scheme I if there exists a constant* $M < \infty$ *(dependent on the differential equation only) such that*

$$|y_m - \hat{y}_m| < M|y_n - \hat{y}_n|$$

for all $0 \le t_n < t_m \le T$, *where* y_i *and* $\hat{y}_i$ *are two numerical solutions.*

Definition 2.4

A method is convergent with respect to a step selection scheme θ *and a formula selection scheme I if the computed solution* y_n *converges to* $y(t_n)$ *for any* $0 \le t_n \le T$ *as* $h \to 0$ *and the starting errors* $\to 0$.

Definition 2.5

A method satisfies the stability condition with respect to a step selection scheme θ *and a formula selection scheme I if the matrices*

$$\hat{T}_n^m = \hat{T}_{m-1}\,\hat{T}_{m-2}\cdots\hat{T}_n$$

are uniformly bounded, where $\hat{T}_n$ *is the value of* T_n *evaluated for the differential equation* $y' = 0$.

Theorem 2.1

If a method is stable with respect to θ *and I, it satisfies the stability condition with respect to* θ *and I.*

The proof is identical to the proof of Theorem 1.

In the discussion below, we will usually refer to (k+1) x (k+1) matrices and (k+1) element vectors. Any vector norm and subordinate matrix norm can be used. When a matrix is partitioned and a norm of one of the partitions is used, that norm is defined as the norm of the (k+1) x (k+1) matrix obtained by replacing all other partitions by 0. Thus, if X is a (k+1) x (k+1) matrix, and

$$\begin{bmatrix} X \end{bmatrix} = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right],$$

then

$$||B|| = \left|\left|\left[\begin{array}{c|c} 0 & B \\ \hline 0 & 0 \end{array}\right]\right|\right|.$$

Thus, $||X|| \leq ||A|| + ||B|| + ||C|| + ||D||$.

The same definition is used for the norm of a part of a vector.

3. ADAMS TYPE METHODS

As with fixed order methods, varying order methods appear to be less stable with the interpolation technique than with the variable step technique. We will prove that a class of formulas which include Adams formulas are stable with respect to any formula and step selection scheme if the variable step technique is used. However, we can only prove that Adams formula is stable for the interpolation technique if the step and order are periodically held constant for p steps after a change to an r-step formula, where $p = r$ if the O_n's are the identity, and $p = r+1$ if the difference method is used.

The next theorem is a straightforward extension of Theorem 2.

<u>*Theorem 2.2*</u>

If a method satisfies the stability condition with respect to θ *and I, if the quantities* α, α^*, β *and* β^* *or* A_n *and* ℓ_n *defined in* [1] *are uniformly bounded, if the* O_{ij} *are bounded and if the truncation error satisfies*

$$||\underline{d}_n|| \leq h_n \, e(h) \qquad \forall_n ,$$

where $e(h) \to 0$ *as* $h \to 0$, *then the method is stable and convergent, and there exist constants* k_3 *and* k_4 *such that*

$$||\underline{\varepsilon}_N|| \leq k_3 \, ||\underline{\varepsilon}_0|| + k_4 \, e(h)$$

for $0 \leq t_N \leq T$.

Theorem 2.3

If a variable step method uses the corrector

$$y_{n+1} = \alpha_1^* y_n + \ldots + \alpha_k^* y_{n-k+1}$$

$$+ \beta_{0,n}^* h_n y'_{n+1} + \ldots + \beta_{m,n}^* h_{n-m} y'_{n-m+1},$$

where the α_i^* *are independent of the step sizes and formula used (we call these "constant-*ρ*" methods), if the* $\beta_{i,n}^*$ *are uniformly bounded, and if* $\rho(\xi) = -\xi^k + \alpha_1^* \xi^{k-1} + \ldots + \alpha_k^* \xi^0$ *satisfies the stability condition, then the method satisfies the stability condition with respect to any step and formula selection schemes.*

NOTE: If an explicit method is used, the predictor coefficients α_i must satisfy the constant-ρ and stability conditions.

Proof

From equation (23)

$$\hat{S}_n = (I - \underline{\ell}_n \underline{e}_d^T)^M A_n$$

$$= \left[\begin{array}{cccc:cccc}
\alpha_1^* & \cdots & \alpha_{k-1}^* & \alpha_k^* & \beta_{1,n}^* & \cdots & \beta_{m-1,n}^* & \beta_{m,n}^* \\
1 & & 0 & 0 & & 0 & & \\
 & \ddots & & & & & & \\
0 & & 1 & 0 & & & & \\
\hdashline
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
 & & & & 1 & & & \\
 & & & & & \ddots & & \\
 & & & & & & 1 & 0
\end{array}\right] = \left[\begin{array}{c:c} C & D_n \\ \hdashline 0 & L \end{array}\right].$$

Since the $||\hat{S}_n||$ are uniformly bounded, the $||\hat{S}_n^{n+p}||$ are uniformly bounded for $0 \leq p \leq m$. From the structure of $\hat{S}_n$, we see that $L^m = 0$; hence if $p > m$,

$$\hat{S}_n^{n+p} = \left[\begin{array}{c:l} C^p & C^{p-m}(C^{m-1} D_n + C^{m-2} D_{n+1} L + \ldots + C^0 D_{n+m-1} L^{m-1}) \\ \hdashline 0 & 0 \end{array}\right].$$

Hence

$$||\hat{S}_n^{n+p}|| \leq ||C^p|| + ||C^{p-m}||\ ||C^{m-1} D_n + \ldots + D_{n+m-1} L^{m-1}||$$

for $p > m$. However, C is the companion matrix of the polynomial $\rho(\xi)$. Hence, if $\rho(\xi)$ satisfies the stability condition, $||C^p||$ and $||C^{p-m}||$ are bounded. Hence $||\hat{S}_n^{n+p}||$ is bounded for all $p \geq 0$.

Q.E.D.

Corollary

If the β_i and β_i^* are chosen to achieve maximum order (at least m for the predictor, $m+1$ for the corrector), the method is stable and

convergent for any step selection scheme, even if m is changed from step to step, and from predictor to corrector, as long as m is bounded, $\rho(\xi)$ satisfies the stability condition, $\rho(1) = 0$, and $m \geq 0$.

NOTE: This means that the variable step, variable order Adams method is stable for any step and order selection schemes.

Proof

Following the argument used in Theorem 4, the $\beta^*_{i,n}$ are solutions of the equations

$$\sum_{i=0}^{m} \beta^*_{i,n} (\omega_i - \omega_{i+1})\omega_i^{r-1} = -\frac{1}{r} \sum_{i=1}^{k} \alpha^*_i \omega_i^r, \qquad 1 \leq r \leq m+1,$$

(or $1 \leq r \leq m$ if $\beta^*_{0,n} = 0$ for an explicit method). Hence, the solutions $\beta^*_{i,n}$ (and $\beta_{i,n}$ for an explicit method) can be bounded above. Thus, the conditions of the preceding theorem are met, and the method satisfies the stability condition.

The order of the corrector will be at least one if $m \geq 0$; hence the truncation error can be bounded by

$$||\underline{d}_n|| \leq h_n \, e(h),$$

where $e(h) \to 0$ as $h \to 0$, so the conditions of Theorem 2.2 are met. Hence, the method is convergent and stable.

Q.E.D.

Theorem 2.4

If a finite set of either explicit, predictor-corrector, or implicit methods are based on Adams formulas using the interpolation step changing technique, if the order of a corrector is equal to or one larger than the order of its predictor, if the step and formula selection schemes are such that there exists a finite constant M with the property that in any M or more consecutive steps there are at least p fixed steps using a predictor of order r, where p=r if all of the O_n's are the identity, or p=r+1 if the difference method--equations (2.1) and (2.2)--is used, then the methods are stable and convergent with respect to those selection schemes.

Proof

A simple extension of the proof of Theorem 6 will be used to show that the method satisfies the stability criterion. Then Theorem 2.2 shows that it is stable and convergent.

For the equation y' = 0, T_n is given by

$$T_n = [I - \underline{\ell}_n \underline{e}_2^T]^M \; A \; C_n \; O_n = R_n \; C_n \; O_n ,$$

where C_n is uniformly bounded by the definition of a step selection scheme, and O_n is uniformly bounded (it is either I or as given in 2.2). If $0 \leq m \leq M$, $||T_n^{n+m}||$ is uniformly bounded. From the hypothesis of the theorem, we are guaranteed that in any M or more consecutive T_q's, there will be at least p adjacent in which $C_q = O_q = I$ (no change in step or order) and A_q and $\underline{\ell}_q$ are constant

(no change in order). Thus,

$$T_n^{n+m} = T_{q+p}^{n+m} R_q^p T_n^q,$$

where $q-n \leq M$. R_q is given by

$$R_q = \left[\begin{array}{c|c} R & 0 \\ \hline \tilde{R} & 0 \end{array}\right],$$

where R is the (r+1) x (r+1) matrix defined in (43) and $\tilde{R}$ is either all zero--if all O_q are the identity, or contains a unit entry in the top right position--if the order increasing technique uses the A_q given in (2.1). From the properties of R given in [1], it follows that

$$R_q^p \underline{x} = \mu_x \underline{e}_1 .$$

As in Theorem 6,

$$\mu_x \leq C \, ||\underline{x}|| .$$

Since $\underline{e}_1$ is a unit eigen vector of R_q, C_q, O_q, and hence T_q, we have

$$\begin{aligned} ||T_n^{n+m} \underline{z}|| &= ||T_{q+p}^{n+m} R_q^p T_n^q \underline{z}|| \\ &= ||T_{q+p}^{n+m} R_q^p \underline{x}|| \text{ where } \underline{x} = T_n^q \underline{z} \\ &= ||T_{q+p}^{n+m} \mu_x \underline{e}_1|| = |\mu_x| \, ||\underline{e}_1|| = |\mu_x| \leq C \, ||\underline{x}|| \\ &= C \, ||T_n^q \underline{z}|| \leq C[\text{Max} \, ||T_j||]^{q-n} \, ||\underline{z}|| \end{aligned}$$

Hence, for $m > M$,

$$||T_n^{n+m}|| \leq C[\text{Max} \, ||T_j||]^M .$$

Thus, the method satisfies the stability condition with respect to the step and formula selection schemes, and so is stable and convergent.

Q.E.D.

4. GENERAL METHODS

Theorem 7 showed that a fixed formula method was stable with respect to small step sizes for both variable step and interpolation methods. There is no direct, useful, equivalent in the sense that if the class of formula used provides one or a few of each order, there is no such concept as a "small" formula change. The closest similar idea is that infrequent formula changes do not cause instability.

<u>*Theorem 2.5*</u>

If a variable formula method is such that:

(i) each formula F_i is strongly stable,

(ii) each formula has order ≥ 1,

(iii) the order changing operations O_{ij} are exact for the equation $y' = 0$, and $||O_{ij}||$ is bounded,

(iv) the coefficients in the methods M_i are uniformly bounded,

then, for each formula F_i there exists an integer p_i such that the method is stable and convergent with respect to a step selection scheme θ that produces small step changes such that

$$\frac{h_{n+1}}{h_n} = 1 + O(h)$$

and a formula selection scheme I that produces infrequent formula changes in the sense that at least p_i steps are taken without formula change after the formula F_i has been selected.

Proof

We will verify that the hypotheses of Theorem 2.2 are satisfied. Hypothesis (iv) of this theorem assures us that A_n and $\underline{\ell}_n$ are bounded uniformly. Hypothesis (ii) tells us that there exists an $e(h) \to 0$ as $h \to 0$ such that

$$||\underline{d}_n|| \leq h_n \; e(h) \qquad \forall_n$$

so we only need verify the stability condition, namely, that $||\hat{T}_n^m||$ is uniformly bounded. Although $\hat{T}_n$ is independent of the differential equation by definition, it does vary with step size and with order. First, we get rid of the step size variation by proving a slight extension of a result contained in Theorem 7. As there, we decompose $\hat{T}_n$ into $\dot{T}_n$, the operator for constant steps, and $h_n \overline{T}_n$, where $\overline{T}_n$ can be bounded. It follows directly from the proof of Theorem 7 that if $\dot{T}_n^m$ and $\overline{T}_n$ are uniformly bounded, then $\hat{T}_n^m$ is uniformly bounded.

In order to bound $\dot{T}_n^m$, we must look at the structure of the different formulas and step changing operations used. We have

$$\dot{T}_n = \hat{S}_i \; O_n ,$$

where $\hat{S}_i$ depends only on the formula F_i used; that is, it is the amplification matrix for a constant step method applied to $y' = 0$. Since the methods have order ≥ 1, $\hat{S}_i$ has an eigen vector with ones in positions corresponding to entries for y_{n-i} in $\underline{y}_n$ and zeros elsewhere. This eigen vector, say $\underline{x}_1$, corresponds to the eigen value one, and it is simply the statement that the method is exact for $y = 1$, $y' = 0$. Since the formulas are strongly stable, all other eigen values of $\hat{S}_i$ are less than one. $\underline{x}_1$ is also an eigen vector of O_n corresponding to $\lambda = 1$. Although $\dot{T}_n^m$ is a product of different matrices, if the changes are sufficiently infrequent, it has the form

$$\dot{T}_n^m = \hat{S}_{i_1} \ldots \hat{S}_{i_1} \; O_{n_1} \; \hat{S}_{i_2} \ldots \hat{S}_{i_2} \; O_{n_2} \; \hat{S}_{i_3} \ldots \hat{S}_{i_k} \; O_{n_k}$$

We will show that the operator $\dot{T}_n^m$ can be bounded because it does not change $\underline{x}_1$, and eventually decreases all other components if there are enough $\hat{S}_i$'s of the same index next to each other. This is shown in Lemma 2.1 below (in more generality than needed at the moment).

Q.E.D.

Lemma 2.1

Let $\{S_i\}$ and $\{O_i\}$ be sets of q x q matrices with the following properties:

(i) If $\{\lambda_{ij}\}$ $j = 1,\ldots, q$, are the eigen values of S_i, then there exist constants ν and η such that

$1 = |\lambda_{i,1}| = |\lambda_{i,2}| =\ldots= |\lambda_{i,\nu}|$,

$1 > \eta \geq |\lambda_{i,\nu+1}|,\ldots, |\lambda_{i,q}|$, $\forall$ i.

(ii) There exists a constant linearly independent set of vectors $\underline{x}_1, \underline{x}_2,\ldots, \underline{x}_\nu$ of norm 1 such that

$S_i \underline{x}_j = \lambda_{i,j} \underline{x}_j$, $j = 1,\ldots, \nu$ and $\forall$ i.

(iii) If Q_i is such that its columns have a norm of one and $Q_i^{-1} S_i Q_i = J_i$ is the Jordan canonical form of S_i (without loss of generality, we can assume that the first ν columns of Q_i are $\underline{x}_1,\ldots, \underline{x}_\nu$), then there exist constants C_1, C_2 such that

$$||Q_i^{-1}|| \leq C_1, \ ||Q_i|| \leq C_2 \ \forall \text{ i}.$$

(iv) $O_i \underline{x}_j = \underline{x}_j$, $j=1,\ldots,\nu$ and $\forall$ i, and there exists a constant C_0 such that $||O_i|| \leq C_0$ $\forall$ i.

Then for any constant $K > C_0 C_1 C_2$ there exists a set of integers $\{p_i\}$ such that

$$||\prod_{j=1}^{N} [(S_j)^{\overline{p}_j} O_j]|| \leq K \qquad \forall N \geq 0$$

whenever $\overline{p}_j \geq p_j$.

Proof

Hypotheses (ii) and (iii) imply that J_i has the form

$$\left[\begin{array}{c|c} \begin{matrix}\lambda_{i1} & & \\ & \ddots & \\ & & \lambda_{i\nu}\end{matrix} & 0 \\ \hline 0 & \begin{matrix}\text{Jordan blocks for} \\ \text{eigenvalues of} \\ \text{magnitude} \leq \eta\end{matrix} \end{array}\right].$$

Hence, $S_i^p = Q_i J_i^p Q_i^{-1}$

$$= Q_i \left[\begin{array}{c|c} \begin{matrix}\lambda_{i1}^p & & \\ & \ddots & \\ & & \lambda_{i\nu}^p\end{matrix} & 0 \\ \hline 0 & U_{ip} \end{array}\right] Q_i^{-1} = Q_i \left[\begin{array}{c|c} \Lambda_i^p & 0 \\ \hline 0 & U_{ip} \end{array}\right] Q_i^{-1},$$

where, from hypothesis (i),

$$|\lambda_{ij}^p| = 1, \quad 1 \leq j \leq \nu,$$

and the elements of U_{ip} are bounded by terms of the form $\eta^{p-j}\binom{p}{j}$ for $0 \leq j \leq q-\nu-1$ and $p \geq j$. (These arise when a $(j+1) \times (j+1)$ Jordan block is raised to the power p.) Hence, p can be chosen to make the norm of U_{ip} as small as desired.

Because the first ν columns of Q_i are independent of i--hypotheses (ii) and (iii)--and because these columns are unit eigen vectors of O_i,

$$Q_i^{-1}\, O_i\, Q_{i+1} = \left[\begin{array}{c|c} I & P_{i1} \\ \hline 0 & P_{i2} \end{array}\right]$$

and from hypotheses (iii) and (iv)

$$||P_{i1}|| \leq C_0\, C_1\, C_2,$$

$$||P_{12}|| \leq C_0\, C_1\, C_2.$$

(Note that $C_0 \geq 1$ because O_i has a unit eigen value and that $C_1\, C_2 \geq 1$ because $1 = ||Q_i\, Q_i^{-1}|| \leq C_1\, C_2$.) By expansion

$$M = \prod_{i=1}^{N} [(S_i)^{\overline{p}_i}\, O_i]$$

$$= Q_1\, J_1^{\overline{p}_1}\, Q_1^{-1}\, O_1\, Q_2\, J_2^{\overline{p}_2} \cdots J_{N-1}^{\overline{p}_{N-1}}\, Q_{N-1}^{-1}\, O_{N-1}\, Q_N\, J_N^{\overline{p}_N}\, Q_N^{-1}\, O_N$$

$$= Q_1 \left[\begin{array}{c|c} \prod_{i=1}^{N} \Lambda_i^{\overline{p}_i} & \sum_{s=1}^{N-1} \left(\prod_{i=1}^{s} \Lambda_i^{\overline{p}_i}\right) P_{s1} \left(\prod_{i=s+1}^{N-1} U_{i\overline{p}_i}\, P_{i2}\right) U_{N\overline{p}_N} \\ \hline 0 & \prod_{i=1}^{N-1} (U_{i\overline{p}_i}\, P_{i2})\, U_{N\overline{p}_N} \end{array}\right] Q_N^{-1}\, O_N.$$

Λ_i^p is a diagonal matrix with unit length elements, hence $||\Lambda_i^p|| = 1$.

If we choose p_i such that

$$||U_{ip}|| \leq \frac{1}{C_0 C_1 C_2} - \frac{1}{K} = C_3,$$

whenever $p \geq p_i$, we have

$$||M|| \le ||Q_1|| \; ||Q_N^{-1}|| \; ||O_N|| \; [1 + \sum_{s=1}^{N-1} C_0C_1C_2 \, (C_3C_0C_1C_2)^{N-1-s} \; C_3 + (C_0C_1C_2C_3)^{N-1} \; C_3]$$

$$\le C_0C_1C_2[1 + \frac{C_0C_1C_2C_3-(C_0C_1C_2C_3)^N}{1-C_0C_1C_2C_3} + (C_0C_1C_2C_3)^N]$$

$$\le C_0C_1C_2[1 + \frac{C_0C_1C_2C_3}{1-C_0C_1C_2C_3}]$$

$$= \frac{C_0C_1C_2}{1-C_0C_1C_2C_3}$$

$$= K$$

as required.

Q.E.D.

Lemma 2.1 allows the matrices $\hat{S}_n$ to have more than one eigen value on the unit circle, but they must have the same set of eigen vectors and simple elementary divisors for those values. Thus, weakly stable methods could be considered. However, the restriction that the eigen vectors match is usually difficult to achieve for other than the principle eigen value $\lambda = 1$. If the eigen values do not match, instability can occur as shown in

Example B

Consider the two weakly stable methods based on

F_1: $\quad y_{n+1} = y_{n-1} + \beta_{1n} \, h_{n-1} \, y'_n + \ldots,$

F_2: $\quad y_{n+1} = y_n - y_{n-1} + y_{n-2} + \beta_{1n} \, h_{n-1} \, y'_n + \ldots.$

(The β_{in} can be picked to get any order desired.) The vector of saved values $\underline{y}_n$ will be

$$\underline{y}_n = [y_n, y_{n-1}, y_{n-2}, h_{n-1} y_n', \ldots]^T,$$

where the maximum number of derivatives used in the two formulas must be saved. The corresponding amplification matrices are

$$S_1 = \left[\begin{array}{ccc|c} 0 & 1 & 0 & \\ 1 & 0 & 0 & \\ 0 & 1 & 0 & \\ \hline & 0 & & \\ & & & \end{array}\right],$$

$$S_2 = \left[\begin{array}{ccc|c} 1 & -1 & 1 & \\ 1 & 0 & 0 & \\ 0 & 1 & 0 & \\ \hline & 0 & & \\ & & & \end{array}\right].$$

If the top lefthand 3 x 3 block blows up, we will not have a stable method, so we will discard the rest. Then we can show

$$S_1^{2k+2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad k \geq 0,$$

$$S_2^{4k+1} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad k \geq 0.$$

If we use a simple formula changing process such that $O_{ij} = I$, then we can consider

$$S_2^{4k+1} S_1^{2\ell+2} = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad k, \ell \geq 0.$$

Since

$$[S_2^{4k+2} S_1^{2\ell+2}]^N = \begin{bmatrix} N+1 & -N & 0 \\ N & -N+1 & 0 \\ N-1 & -N+2 & 0 \end{bmatrix}, \qquad N \geq 1, \qquad k, \ell \geq 0$$

it is evident that no matter how large k and ℓ are, the product will still be unbounded.

Theorem 2.5 states that some integers p_i do exist, but does not give a very useful constructive way of finding the smallest values. However, we can show by example that these integers are greater than one in some cases.

Example C

Consider methods based on the formulas

$$F_1: \quad y_{n+1} = 2y_n - \frac{5}{4} y_{n-1} + \frac{1}{4} y_{n-2} + \beta_{0,n} h_n y'_{n+1} + \ldots,$$

$$F_2: \quad y_{n+1} = \frac{3}{4} y_{n-1} + \frac{1}{4} y_{n-2} + \beta_{0,n} h_n y'_{n+1} + \ldots.$$

Both of these formulas are strongly stable. (The roots of F_1 and F_2 are 1, +1/2, +1/2, and 1, -1/2, -1/2, respectively.) If $\underline{y}_n = [y_n, y_{n-1}, y_{n-2}, \ldots]^T$ and if we ignore the derivative terms by considering the equation $y' = 0$, we find

$$S_1 = \begin{bmatrix} 2 & -5/4 & 1/4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$S_2 = \begin{bmatrix} 0 & 3/4 & 1/4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Hence,

$$S_1S_2 = \begin{bmatrix} -5/4 & 7/4 & 2/4 \\ 0 & 3/4 & 1/4 \\ 1 & 0 & 0 \end{bmatrix}.$$

The eigen values of S_1S_2 are 1, and $-3/4 \pm \sqrt{\frac{1}{2}}$. Since $\left|-\frac{3}{4} - \sqrt{\frac{1}{2}}\right| \doteqdot 1.46$ exceeds one, $(S_1S_2)^N$ is unbounded, so alternating formulas each step will lead to instability.

The p_i were calculated for the backward differentiation formulas using the interpolation step changing technique with the $0_{ij} = I$. Let S_i be the stability matrix for the i-th order method and let

$$Q_iJ_iQ_i^{-1} = S_i, \quad i = 1, \ldots, 6.$$

The Q_i and J_i are partitioned as follows:

$$Q_i = \left[\begin{array}{c|cccccc} 1 & x & x & x & x & x & x \\ \hline 0 & & & & & & \\ 0 & & & & & & \\ 0 & & & \tilde{Q}_i & & & \\ 0 & & & & & & \\ 0 & & & & & & \\ 0 & & & & & & \end{array}\right], \quad J_i = \left[\begin{array}{c|cccccc} 1 & x & x & x & x & x & x \\ \hline 0 & & & & & & \\ 0 & & & & & & \\ 0 & & & \tilde{J}_i & & & \\ 0 & & & & & & \\ 0 & & & & & & \\ 0 & & & & & & \end{array}\right].$$

Each Q_i is chosen so that the columns corresponding to the zero eigenvalues λ_{ij} of S_i induced by its extension from (i+1)-space to 7-space are of the form $\underline{e}_j$. Q_i^{-1} is similarly partitioned to yield $\tilde{Q}_i^{-1}$. The p_{ij} are then chosen as the smallest integers satisfying

$$||\tilde{Q}_i^{-1}\tilde{Q}_j\tilde{J}_j^{p_{ij}}||_\infty < 1.$$

Each p_{ij} is the number of constant order steps required after a change from order i to order j. The table of p_{ij} below was calculated by Mr. Mike Ostrar.

i \ j	1	2	3	4	5	6
1	1	1	1	2	3	6
2	1	1	1	2	3	6
3	1	1	1	2	4	9
4	1	1	1	1	4	11
5	1	1	1	1	1	5
6	1	1	1	1	2	1

If an order change of at most one is used, the number of steps p_i needed after a change to order i is not more than i-1.

Although we did not calculate similar tables for the interpolation technique with differencing or the variable step technique, we believe the tables will be similar.

5. CONCLUSIONS

Again, on limited evidence, the variable step technique seems to be more stable than the interpolation technique when formula changing is considered, but it does seem valuable to restrict the frequency with which the formulas are switched in all but Adams methods. Otherwise, the recommendations of reference [1] are unchanged.

LIST OF REFERENCES

[1] Gear, C. W. and Tu, K. W., "The Effect of Variable Mesh Size on the Stability of Multistep Methods," submitted to the SIAM Journal on Numerical Analysis, 1972; Department of Computer Science Report No. 570, University of Illinois at Urbana-Champaign, 1973.

[2] Shampine, L. F. and Gordon, M. K., "Local Error and Variable Order, Variable Step Adams Codes," to appear, SIAM Journal on Numerical Analysis.

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The variable step Adam's method is shown to be stable for any order changing scheme. The Nordsieck form of Adam's method, however, is shown to be stable only if the step size and order are fixed for p steps following a change to an r-step method, where p is r or r+1 depending on the algorithm used to interpolate the necessary higher derivatives. Finally, general consistent and strongly stable multistep and multivalue methods are shown to be stable if the method is fixed for a certain number of steps following each method change and step size changes are small. This number is independent of the differential equation and the step sizes.

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